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Analytical Solution of the advection diffusion equation using Fourier and square complement methods and comparing with Gaussian model

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Abstract

Two approaches have been used to solve the two-dimensional atmospheric diffusion problem analytically. First, the integration has been solved using the separation of variables, Fourier transform, and square complement methods. Next, the normalized cross-wind integrated concentration of pollutants at the earth's surface with variable eddy diffusivity and wind speed with power in vertical height and downwind distance has been obtained using the Gaussian plume model. creating a Gaussian plume model by using the dispersion parameters of Briggs and Brookhaven National Laboratories. Sulfur Hexafluoride (SF_6) tracer data sets from the northern portion of Copenhagen, Denmark, and Iodine-135 data sets under unstable conditions, are used to validate the current and Gaussian models.

Keywords: Fourier transform, and square complement methods; Sulfur Hexafluoride (SF_6); Iodine-135

Introduction

The advection-diffusion equation is often only analytically solved for stationary conditions and under the strict assumption of wind speed profiles (U) and eddy diffusivity coefficients (K). Throughout the whole ABL, they are presumed to be constant or to follow a power law[1-5]. The advection-diffusion equation was solved by[6] Moriera et al. (2005) using the Laplace transform while taking into account the ABL as a multilayer system.

Numerous integral transforms are commonly employed as a means of resolving a wide range of scientific issues. Fourier transforms, Mellin transforms, and Hankel transforms are all highly helpful in many real-world applications. The case for functions of two variables mainly to understand an origin for a Hankel transform was studied by[7-8]. Weather and topography have an impact on the movement, dispersion, and deposition of air pollutants. The movement of pollutants in a turbulent atmosphere has long been described by the atmospheric advection-diffusion equation, which was examined by[3] An analytical solution that was fundamentally important and described with physical facts was explored[2]. Analytical solutions to assess the precision and effectiveness of the numerical answers was employed[9]. For the simulation of atmospheric diffusion issues under steady conditions, The performance of a unified formal analytical solution was investigated[10].

Basic fractional differential equation models for the steady state geographic distribution of a non-reactive pollutant's concentration in the PBLwas examined by[11] . Compared to the conventional Gaussian model, they discovered that fractional derivatives models perform better.

An analytical dispersion model for sources in the air surface layer with dry deposition to the ground surface was examined by[12]. Additionally,the impact of eddy diffusivity change on the behavior of the advection-diffusion equation was investigated by[13]. The diffusion from two dimensions time-dependent and three dimensions of diffusion equation under various stability situations was studied by[14]. The World Meteorological Organization (WMO) has examined the atmospheric diffusion models that are commonly used for regulatory purposes. The magnitude and location of the highest ground level concentration are the most significant characteristics that these models anticipate. More regulatory applications use the Gaussian Plume Model (GPM) than any other dispersion model.

The present study presents an analytical solution of the two-dimensional atmospheric diffusion equation, which is first solved using the separation of variables, Fourier transform, and square complement methods. The Gaussian plume model is then used to obtain the normalized cross-wind integrated concentration of pollutants at the earth's surface, where the variables of wind speed and eddy diffusivity are takien as functions of the power law of downwind distance and vertical height. Gaussian plume model obtained through national laboratory work by Briggs and Brookhaven. The observed data sets of the tracer sulfur hexafluoride (SF_6) and iodine-135[15] ,under unstable conditions, at the Northern portion of Copenhagen, Denmark[16] are used to validate the anticipated and Gaussian models.

Mathematical Model

1-First model

The diffusion equation of pollutants in air can be written in the form by[17] as follows:

$$u\frac{\partial C(x,y,z)}{\partial x} = \frac{\partial}{\partial y}\left(K_y\frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial y}\left(K_z\frac{\partial C}{\partial z}\right) \tag{1}$$

The concentration in the three dimensions is denoted by C(x,y,z), the mean wind orientated in the x direction is denoted by u, and the crosswind and vertical turbulent eddy diffusivity coefficients of the Planetary Boundary Layer (PBL) are denoted by K_y and K_z, respectively. These boundary conditions are applied to Eq. (1).

The flux equals zero at the ground surface and the top of PBL as follows:

$$K_z\frac{\partial C}{\partial z} = 0 \qquad at\ z = 0 \tag{i}$$

$$K_z\frac{\partial C}{\partial z} = 0 \qquad at\ z = h \tag{ii}$$

The mass continuity is verified as follows:

$$C(0,z) = \frac{Q}{U}\delta(z - h_s) \qquad at\ x = 0 \tag{iii}$$

where δ is the Dirac delta function, h_s is the stack height, h is the PBL height, and Q is the emission rate. One obtains the following by integrating with regard to y from -∞ to ∞:

$$u\frac{\partial}{\partial x}\int_{-\infty}^{\infty} C(x,y,z)dy = K_y\frac{\partial C(x,y,z)}{\partial y}\bigg|_{-\infty}^{\infty} + \frac{\partial}{\partial z}\left(K_z\frac{\partial}{\partial z}\right)\int_{-\infty}^{\infty} C(x,y,z)dy \tag{2}$$

Suppose that:

$$\int_{-\infty}^{\infty} C(x,y,z)dy = C_y(x,z) \tag{3}$$

Since
$$K_y \frac{\partial C(x,y,z)}{\partial y}\bigg|_{-\infty}^{\infty} = 0 \tag{4}$$

By substituting from equations (3) and (4) into equation (2), one can get

$$u\frac{\partial C_y(x,z)}{\partial x} = \frac{\partial}{\partial z}\left(K_z \frac{\partial C_y(x,z)}{\partial z}\right) \tag{5}$$

Keeping in mind that the PBL's height is divided into N sub-intervals so that u and K_z take on average values within each interval[18]. The average values are then as follows: Equation (5) can only be solved by solving "N" problems of the type

$$K_n = \frac{1}{z_{n+1}-z_n}\int_{z_{n+1}}^{z_n} K_z(z)dz \tag{6}$$

$$u_n = \frac{1}{z_{n+1}-z_n}\int_{z_{n+1}}^{z_n} u(z)dz \tag{7}$$

$$u_n\frac{\partial C_y(x,z)}{\partial x} = K_n \frac{\partial}{\partial z}\left(\frac{\partial C_y(x,z)}{\partial z}\right) \tag{8}$$

C_y (x, z) is called cross-wind integrated concentration of n^{th} sub-interval. Let the solution of equation (8) using separation variables is in the form.

$$C_y(x,z,h) = S(x)R(z,h) \tag{9}$$

By substituting from Eq. (9) in Eq. (8), one gets:

$$u_n R(z,h)\frac{dS}{dx} = K_n S(x)\frac{d^2R}{dz^2} \tag{10}$$

Divide both sides over S(x) R(z, h)

$$\frac{u_n}{S(x)}\frac{dS}{dx} = \frac{K_n}{R(z,h)}\frac{d^2R}{dz^2}\text{-} \tag{11}$$

Let's equal both sides to the constant $-p^2$

$$\frac{1}{S(x)}\frac{ds}{dx} = -p^2$$

$$\frac{dS}{S(x)} = -\int P^2 dx$$

$$Ln\, S(x) = -P^2 x + Ln\, g$$

where, g is a constant.

$$Ln \frac{S(x)}{g} = -P^2 x$$

$$S(x) = g e^{-P^2 x} \tag{12}$$

$$\frac{K_n(z)}{u_n R(z,h)} \frac{d^2 R}{dz^2} = -P^2$$

$$\frac{d^2 R}{dz^2} = \frac{-u_n P^2}{K_n} R(z,h) \tag{13}$$

Then the solution of Eq. (13) is in the form:

$$R(z,h) = A(h) e^{iPz\sqrt{\frac{u_n}{K_n}}} + B(h) e^{-iPz\sqrt{\frac{u_n}{K_n}}} \tag{14}$$

Where A(h) and B(h) are constants depend on the mixing height "h". The, one can write Eq. (9) in the form:

$$C_y(x,z,h) = gA(h) e^{-P^2 x + iPz\sqrt{\frac{u_n}{K_n}}} + gB(h) e^{-P^2 x - iPz\sqrt{\frac{u_n}{K_n}}} \tag{15}$$

Each term in this equation is oscillatory but bounded as z→ $\pm\infty$ for all distances $x \geq 0$.

$$\lim_{z \to \pm\infty} gA(h) e^{-P^2 x + iPz\sqrt{\frac{u_n}{K_n}}} = oscillatory$$

$$\lim_{z \to \pm\infty} \left| gA(h) e^{-P^2 x + iPz\sqrt{\frac{u_n}{K_n}}} \right| = gA(h) e^{-P^2 x} \tag{16}$$

Also

$$\lim_{z \to \pm\infty} \left| gB(h) e^{-P^2 x - iPz\sqrt{\frac{u_n}{K_n}}} \right| = gB(h) e^{-P^2 x} \tag{17}$$

Since $0 < P < \infty$ varies continuously the sum these solutions depend on the integration of P. So, the general solutions:

$$C_y(x,z,h) = \int_0^\infty g(P) A(P,h) e^{-P^2 x + iPz\sqrt{\frac{u_n}{K_n}}} dP + \int_{-\infty}^0 g(-P) B(-P,h) e^{-P^2 x - iPz\sqrt{\frac{u_n}{K_n}}} dP \tag{18}$$

Also, we can write Eq. (18) in the form:

$$C_y(x,z,h) = \int_{-\infty}^\infty [g(P)A(P,h) + g(-P)A(-P,h)]\, e^{-P^2 x + iPz\sqrt{\frac{u_n}{K_n}}} dP \tag{19}$$

Let

$$V(P,h) = g(P)A(P,h) + g(-P)A(-P,h)$$

$v(P,h) = g(P)\,A(P,h) \qquad if \quad P > 0$

$v(P,h) = g(-P)\,A(-P,h) \qquad if \quad P < 0$

Eq. (19) becomes:

$$C_y(x,z,h) = \int_{-\infty}^{\infty} V(P,h) e^{-P^2 x + iPz\sqrt{\frac{u_n}{K_n}}} dP \tag{20}$$

To find the value of V(P, h) use the Fourier transform of $\delta(z-h)$ as follows:

$$\hat{g}(P) = \int_{-\infty}^{\infty} \delta(z-h) e^{-iPz\sqrt{\frac{u_n}{K_n}}} \tag{21}$$

$$\hat{g}(P) = e^{-iph\sqrt{\frac{u_n}{K_n}}} \tag{22}$$

Let, $P\sqrt{\frac{u_n}{K_n}}$=w and $dP = dw\sqrt{\frac{K_n}{u_n}}$

$$\hat{g}(P) = e^{-iwh} \tag{23}$$

The inverse Fourier transfer of $\delta(z-h)$ is:

$$\delta(z-h) = \int_{-\infty}^{\infty} e^{-iwh}\, e^{+iwz} \sqrt{\frac{K_n}{u_n}} \frac{dw}{2\pi} \tag{24}$$

$$\delta(z-h) = \int_{-\infty}^{\infty} e^{iw(z-h)} \sqrt{\frac{K_n}{u_n}} \frac{dw}{2\pi} \tag{25}$$

By using the third condition of the boundary condition Eq. (iii) at x=0, then Eq. (20) becomes in the form:

$$\delta(z-h) = \frac{u_n}{Q} \int_{-\infty}^{\infty} V(P,h)\, e^{iwz} \sqrt{\frac{K_n}{u_n}}\, dw \tag{26}$$

By substituting $\delta(z-h)$ with the value in Eq. (25), we find that:

$$\int_{-\infty}^{\infty} e^{iw(z-h)} \sqrt{\frac{K_n}{u_n}} \frac{dw}{2\pi} = \frac{u_n}{Q} \int_{-\infty}^{\infty} V(P,h)\, e^{iwz} \sqrt{\frac{K_n}{u_n}}\, dw \tag{27}$$

So

$$V(P,h) = \frac{Q}{2\pi u_n} e^{-iwh}$$

$$V(P,h) = \frac{Q}{2\pi u_n} e^{-iPh\sqrt{\frac{u_n}{K_n}}} \tag{28}$$

Then, Eq. (20) can be written in the form:

$$C_y(x,z,h) = \frac{Q}{u_n}\int_{-\infty}^{\infty} e^{-P^2x+iP(z-h)\sqrt{\frac{u_n}{K_n}}}\frac{dP}{2\pi} \tag{29}$$

By using square compliment method to solve the integration:

$$P^2x - iP(z-h)\sqrt{\frac{u_n}{K_n}} = x\left[P^2 - \frac{iP}{x}(z-h)\sqrt{\frac{u_n}{K_n}}\right]$$

$$P^2x - iP(z-h)\sqrt{\frac{u_n}{K_n}} = x\left[P^2 - \frac{2ip}{2X}(z-h)\sqrt{\frac{u_n}{K_n}}\right] + \left[\frac{1}{2x}(z-h)\sqrt{\frac{u_n}{K_n}}\right]^2 - \left[\frac{1}{2x}(z-h)\sqrt{\frac{u_n}{K_n}}\right]^2 \tag{30}$$

$$P^2x - iP(z-h)\sqrt{\frac{u_n}{K_n}} = x\left[P - \frac{i(z-h)}{2x}\sqrt{\frac{u_n}{K_n}}\right]^2 - \left[\frac{i(z-h)}{2x}\sqrt{\frac{u_n}{K_n}}\right]^2 \tag{31}$$

$$P^2x - iP(z-h)\sqrt{\frac{u_n}{K_n}} = x\left[\left[P - \frac{i(z-h)}{2x}\sqrt{\frac{u_n}{K_n}}\right]^2 + \left[\frac{(z-h)}{2x}\sqrt{\frac{u_n}{K_n}}\right]^2\right] \tag{32}$$

By substituting this formula into the integral in Eq. (29), one gets:

$$C_y(x,z,h) = \frac{Q}{u_n}\int_{-\infty}^{\infty} e^{\left[-x\left(P-\frac{i(z-h)}{2x}\sqrt{\frac{u_n}{K_n}}\right)^2 - \frac{x(z-h)^2u_n}{4K_nx^2}\right]}\frac{dP}{2\pi} \tag{33}$$

$$C_y(x,z.h) = \frac{Q}{u_n} e^{-\frac{u_n(z-h)^2}{4K_nx}} \int_{-\infty}^{\infty} e^{-x\left(P-\frac{i(z-h)}{2x}\sqrt{\frac{u_n}{K_n}}\right)^2}\frac{dP}{2\pi} \tag{34}$$

Let $\sqrt{x}\left(P - \frac{i(z-h)}{2x}\sqrt{\frac{u_n}{K_n}}\right) = n$

$$dP = \frac{1}{\sqrt{x}}dn \tag{35}$$

$$C_y(x,z.h) = \frac{Q}{u_n} e^{-\frac{u_n(z-h)^2}{4K_nx}} \int_{-\infty}^{\infty} e^{-n^2}\frac{dn}{2\pi\sqrt{x}}$$

$$C_y(x,z.h) = \frac{Q}{2\pi u_n\sqrt{x}} e^{-\frac{u_n(z-h)^2}{4K_nx}} \int_{-\infty}^{\infty} e^{-n^2}dn$$

$$C_y(x,z.h) = \frac{Q}{2u_n\sqrt{\pi x}} e^{-\frac{u_n(z-h)^2}{4K_nx}} \tag{36}$$

The concentration in three dimensions is as follows:

$$C(x,y,z,h) = \frac{Q}{2u_n\pi\sigma_y\sqrt{2x}} e^{-\frac{u_n(z-h)^2}{4K_n x}} e^{-\frac{y^2}{2\sigma_y^2}} e^{-\frac{\nu x}{u_n}} \quad (37)$$

where, σ_y is the standard deviation in y direction and $e^{-\frac{\nu x}{u}}$ is the radioactive decay for the specified nuclide, ν is radioactive coefficient.

Assuming the wind speed u and the vertical eddy diffusivity k_z are taken as power law in vertical distance "z" as follows:

$$u_n = \beta z^p \quad (38)$$

$$K_n = \gamma z^{\epsilon} \quad (39)$$

Where, $\beta = 0.16\left(\frac{w_*}{u}\right)^2$ *and* $\gamma = 0.31\left(\frac{w_*}{u}\right)^2$, w_* is the convective vertical velocity and u is the wind speed at 10 m. p *and* ε are the values which depend on the stability conditions as in table (1). Also, taking

$$K_n = 0.4x \quad (40)$$

Where, x is the downwind distance and the wind speed is taken at reference height 10 m.

2- **Second Metod**

Concentration in a Gaussian model can be written by[19]

$$C(x,y,z) = \frac{Q}{2\pi u\sigma_y\sigma_z} e^{\frac{-y^2}{2\sigma_y^2}} [e^{\frac{-(z-H)^2}{2\sigma_z^2}} + e^{\frac{-(z+H)^2}{2\sigma_z^2}}] e^{-\frac{\nu x}{u}} \quad (41)$$

where y, z are the crosswind and vertical coordinates, respectively; u is the mean wind speed; σ_y and σ_z are the dispersion parameters in the crosswind and vertical directions of the plume; Q is the emission rate; and H is the effective stack height; $H=h_s+\Delta h$, where hs is the stack height and Δh is the plume rise. For an isotope with $\upsilon=2.9\text{x}10^{-5}\ s^{-1}$,

$$H = h_s + \Delta h = h_s + 3(w/u)D \quad (42)$$

where, w is the exit velocity of the pollutants, and D is the internal stack diameter

Because they require stationary and homogeneous turbulence in the PBL, where the flow may be assumed quasi-stationary for appropriate short periods of time (from 10 min to 1 h), it is possible to assume that the mean concentration of a pollutant plume emitted from a point source has a Gaussian distribution, which is highly idealized.

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Taking Briggs and Brookhaven national laboraterly of dispersion parameters of σ_y *and* σ_z as in Tables (1) and (2) as follows:

Table (1). Briggs formulas (1973) for $\sigma_y(x)$ and $\sigma_z(x)$ in urban area.

Stability classes	σ_z (m)	σ_y (m)
A	$0.24x\,(1+0.001x)^{1/2}$	$0.32x\,(1+0.0004x)^{-1/2}$
B	$0.24x\,(1+0.001x)^{1/2}$	$0.32x\,(1+0.0004x)^{-1/2}$
C	$0.20x$	$0.32x\,(1+0.0004x)^{-1/2}$
D	$0.14x\,(1+0.0003x)^{-1/2}$	$0.16x\,(1+0.0004x)^{-1/2}$
E	$0.08x\,(1+0.00015x)^{-1/2}$	$0.11x\,(1+0.0004x)^{-1/2}$
F	$0.08x(1+0.00015x)^{-1/2}$	$0.11x\,(1+0.0004x)^{-1/2}$

Table (2) Brookhaven National Laboratory dispersion parameterization scheme

Stabilityclasses	Values of σ_y	Values of σ_z
A	$\sigma_y = 0.40x^{0.91}$	$\sigma_z = 0.41x^{0.91}$
B	$\sigma_y = 0.40x^{0.91}$	$\sigma_z = 0.41x^{0.91}$
C	$\sigma_y = 0.36x^{0.86}$	$\sigma_z = 0.33x^{0.86}$
D	$\sigma_y = 0.32x^{0.78}$	$\sigma_z = 0.22x^{0.78}$

Results and Discussion:

Eq. (34) with downwind distance and Eq. (38) using dispersion prameters of Briggs and Brookhaven are using to get the predicted concentrations are compared with observed date of sulfur hexafluoride (SF_6) from Copenhagen in Denmark[16] where the stack height "h_s" equals 115m, exist velocity "w" equals 4m/s and the diameter "D" equals 1m as in Table (4) as follows:

Table (3) Power-law exponent p and ϵ of wind speed and eddy diffusivity as a function of air stability in urban area.

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	A	B	C	D	E	F
p	0.15	0.15	0.20	0.25	0.40	0.60
ϵ	0.85	0.85	0.80	0.75	0.60	0.40

Table (4) Comparison between the predicated and observed crosswind integrated normalized concentration at different downwind distance, wind speed and distance for the different runs.

Run No.	Date	PG Stability	h(m)	σ_w (ms^{-1})	U_{10} (ms^{-1})	U_{115} (ms^{-1})	Distance (m)	C_y/Q (10^{-4} sm^{-2})			
								Observed	Proposed model Eq.(37)	GaussBriggs model Eq. (41)	GaussBrook model Eq. (41)
1	20-9-78	A	1980	0.83	2.1	3.03	1900	6.48	2.73	8.20	3.16
1	20-9-78	A	1980	0.83	2.1	3.03	3700	2.31	8.90	1.46	1.00
2	26-9-78	C	1920	1.07	4.9	7.99	2100	5.38	0.12	4.82	2.65
2	26-9-78	C	1920	1.07	4.9	7.99	4200	2.95	1.92	1.06	1.00
3	19-10-78	B	1120	0.68	2.4	3.46	1900	8.20	8.82	7.17	2.76
3	19-10-78	B	1120	0.68	2.4	3.46	3700	6.22	10.99	1.28	0.88
3	19-10-78	B	1120	0.68	2.4	3.46	5400	4.30	10.11	0.46	0.99
5	9-11-78	C	820	0.71	3.1	5.05	2100	6.72	7.53	10.03	4.18
5	9-11-78	C	820	0.71	3.1	5.05	4200	5.84	7.64	2.34	1.58
5	9-11-78	C	820	0.71	3.1	5.05	6100	4.97	6.75	1.00	1.60
6	30-4-78	C	1300	1.33	7.2	11.73	2000	3.96	0.24	4.75	1.18
6	30-4-78	C	1300	1.33	7.2	11.73	4200	2.22	1.84	1.01	1.00
6	30-4-78	C	1300	1.33	7.2	11.73	5900	1.83	2.19	0.47	0.51
7	27-6-78	B	1850	0.87	4.1	5.91	2000	6.70	0.45	3.70	1.55
7	27-6-78	B	1850	0.87	4.1	5.91	4100	3.25	3.51	0.57	0.47
7	27-6-78	B	1850	0.87	4.1	5.91	5300	2.23	4.18	0.28	0.59
8	6-7-78	D	810	0.72	4.2	7.73	1900	4.16	3.48	8.89	2.60
8	6-7-78	D	810	0.72	4.2	7.73	3600	2.02	4.76	5.86	3.79
8	6-7-78	D	810	0.72	4.2	7.73	5300	1.52	4.48	2.80	1.48
9	19-7-78	C	2090	0.98	5.1	8.31	2100	4.58	0.04	4.63	2.55
9	19-7-78	C	2090	0.98	5.1	8.31	4200	3.11	1.45	1.02	0.96
9	19-7-78	C	2090	0.98	5.1	8.31	6000	2.59	2.33	0.45	0.99

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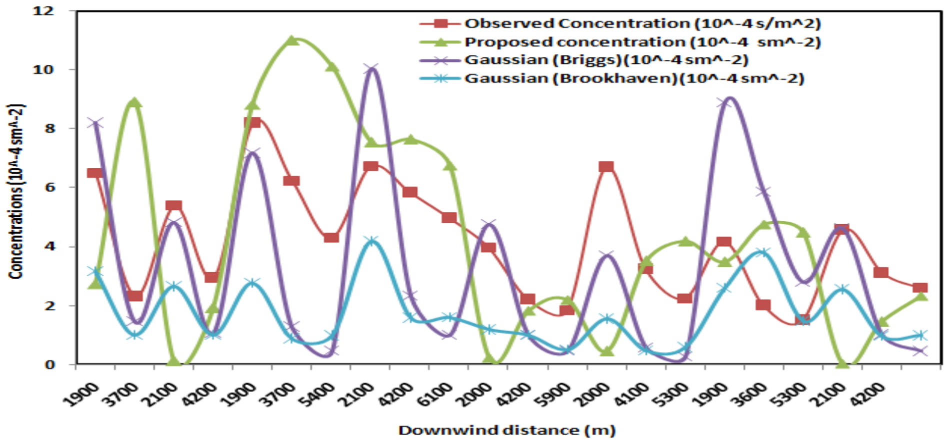

Fig**. (1) Shows that the variation of the concentration with down wind distance with respect to Copenhagen**.

Fig. (1) illustrates that the Propsed and Gaussian plume models using the Briggs approach agree well and are closer to the measured concentrations than the Gaussian plume model using the Brookhaven method. Additionally, as illustrated in Fig. (2), the majority of locations utilizing the Gaussian plume model with Brookhaven dispersions are situated outside a factor of two, whereas the anticipated and Gaussian plume model concentrations using the Briggs model are positioned inside a factor of two with observed data.

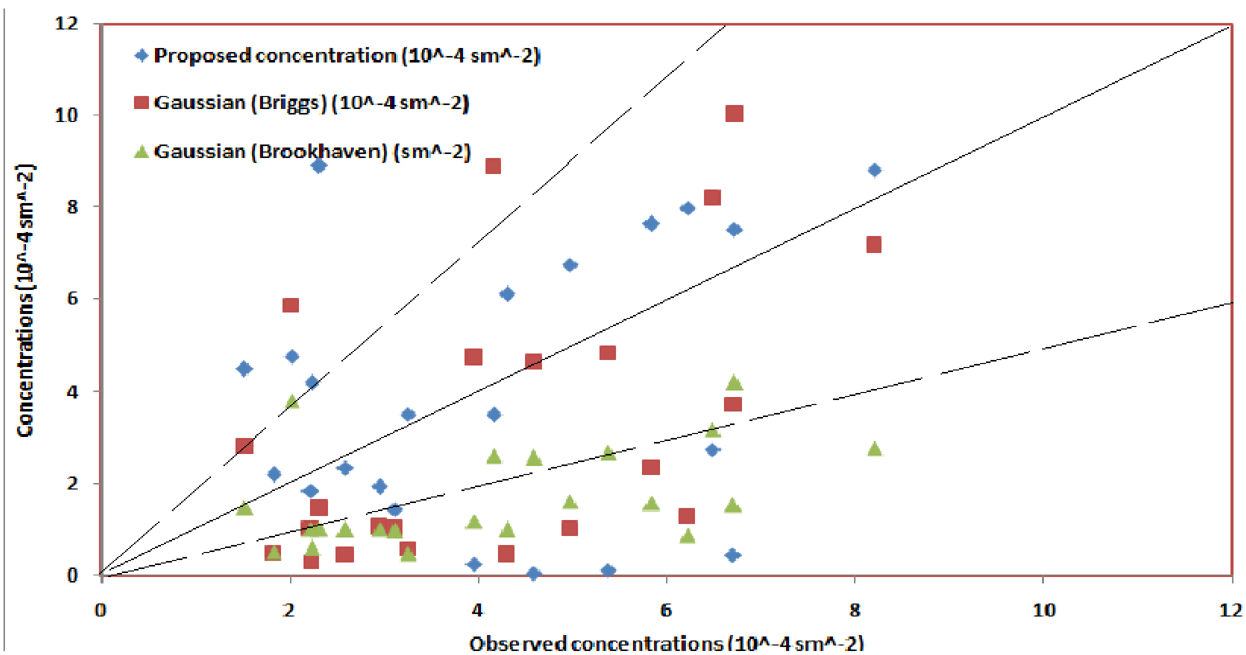

Fig. (2) Shows that the variations of proposed, Gasussian concentions with the observed concentrations

Dispersion studies carried out in unstable settings, yielded the observed concentration of Iodine-135. Below the plume center line, where h_s = 43m, W = 4m/s, and D = 1m, the projected concentrations by Eq. (35) with centerline at ground surface using Eqns (36) and (37) where ε and p are derived from Table (3) and Eq. (38). These are shown in Table (5). Figs (3) and (4) compare the observed and projected amounts of I^{135} in unstable conditions[15] .

Table (5) Comparison between the predicated and observed crosswind- integrated normalized concentration at different downwind distance, wind speed and distance for the different runs.

Run	*Stability class*	*h (m)*	*Wind Direction (deg)*	*U_{10} m (m/s)*	*Q (Bq)*	*Distance (m)*	*Observed C (Bq/m³)*	*Proposed* Model Eq.(37) *(Bq/m³)*	*GaussBrigg* model *Eq. (41) (Bq/m³*	*GaussBrigg* model *Eq. (41) (Bq/m³*
1	A	600.85	301.1	4	1028571	100	0.025	0.029	0.039	0.042
2	A	801.13	278.7	4	1050000	98	0.037	0.027	0.039	0.042
3	B	973	190.2	6	42857.14	115	0.091	0.058	0.070	0.076
4	C	888	197.9	4	471428.6	135	0.197	0.159	0.158	0.316
5	A	921	181.5	4	492857.1	99	0.272	0.273	0.267	0.288
6	D	443	347.3	4	514285.7	184	0.188	0.278	0.243	0.246
7	C	1271	330.8	4	1007143	165	0.447	0.269	0.231	0.486
8	C	1842	187.6	4	1043571	134	0.123	0.167	0.348	0.296
9	A	1642	141.7	4	1033929	96	0.032	0.044	0.042	0.060

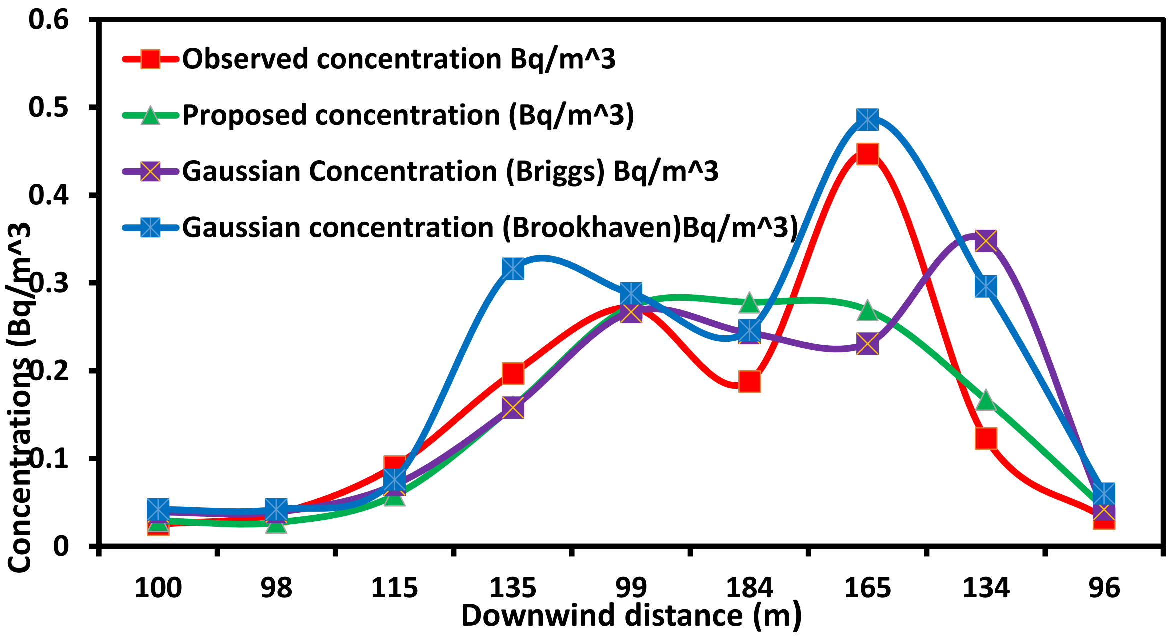

Fig. (3) Shows that the variation of the concentrations with downwind distance.

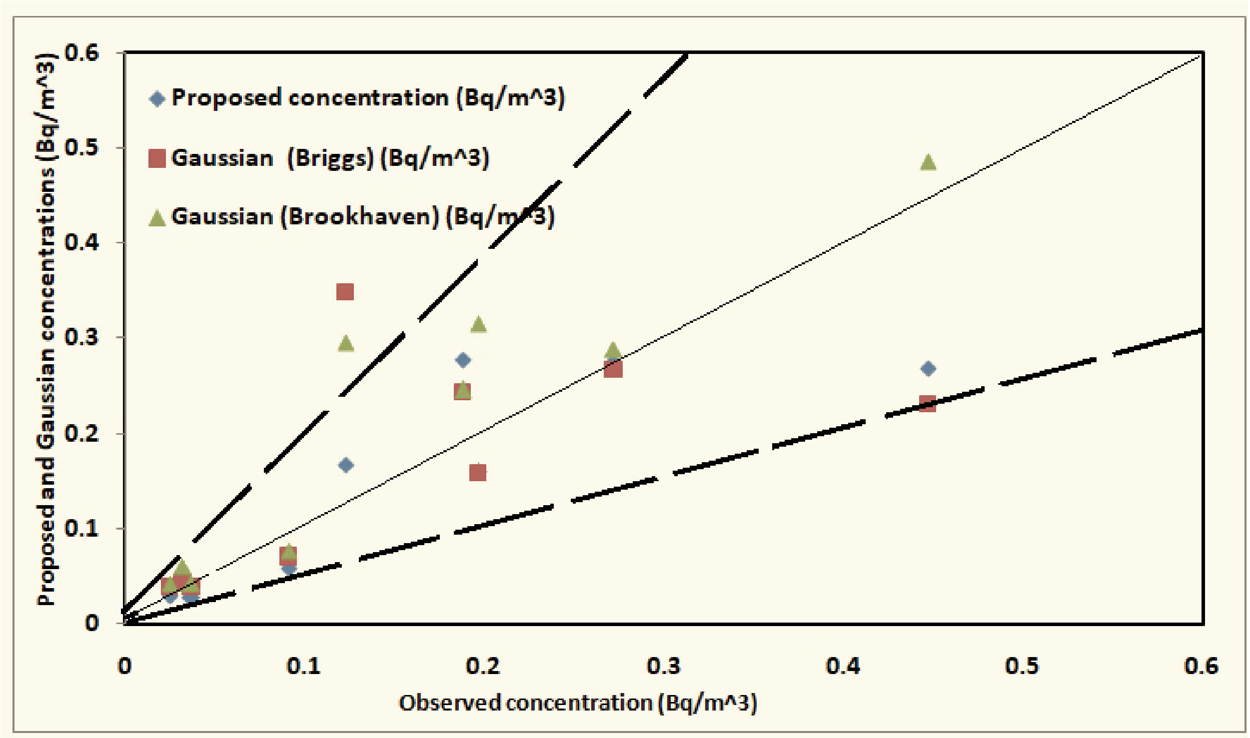

Fig. (4) Shows the variations of proposed and Gaussian plume models with the observed

Additionally, Fig. (3) demonstrates that the Propsed and Gaussian plume models using the Briggs approach agree well and are closer to reported concentrations than the Gaussian plume model using the Brookhaven technique. Additionally, as Fig. (4) illustrates, the concentrations of the suggested Gaussian plume models using the Briggs and Brookhaven methods are within a factor of two of the observed data.

MODEL EVALUATION STATISTICS

One uses the following statistical idiocies to describe the agreement between the predicted and observed concentrations. These are discussed by[20] as follows:

Table (6) depicts a comparison of the concentrations in unstable conditions—predicted,Gaussian and observed Concentrations in Copenhagen in Denmark.

Models	NMSE	FB	COR	FAC2
Predicted	0.63	-0.03	0.28	1.03
Gaussian model (Briggs)	0.53	0.24	0.54	0.79
Gaussian model (Brookhaven)	1.2	0.84	0.48	0.41

Where, NMSE is Normalized Mean Square Error, FB is Fraction Bias, COR is the Correlat – ion Coefficient and FAC2 Factor of Two (FAC2).

According to Table (6), the Gaussian concentraions using the Brookhaven technique are outside of a factor of two, while the anticipated and Gaussian concentraions using the Briggs method are inside a factor of two. Additionally, based on observed SF6 concentrations, the suggested and Gaussian concentrations using the Briggs technique obtained 100% and 0.79%, respectively, whereas the Gaussian concentrations using the Brookhaven method obtained 0.41%.

Table (7) observed and Gaussian Concentrations in unstable condition.

Models	NMSE	FB	COR	FAC2
Predicted	0.05	0.08	0.85	0..92
Gaussian model (Briggs)	0.02	-0.02	0.62	1.02
Gaussian model (Brookhaven)	0.06	-0.27	0.93	1.31

According to Table (7), the observed concentrations of I^{135} are within a factor of two of all the Gaussian concentrations predicted by the Briggs and Brookhaven techniques. Using the Briggs and Brookhaven approaches, all of the suggested Gaussian concentrations are 92%,!00%, and 100% of the reported I^{135} concentrations, respectively. For all suggested and all Gaussian models, NMSE and FB are near zero.

Conclusions

The proposed and the Gaussian plume model using the Briggs approach agree well and are closer to the measured amounts of SF_6 than the Gaussian plume model using the Brookhaven method. Additionally, the majority of sites utilizing the Gaussian plume model with Brookhaven dispersions are outside of a factor of two with regard to SF_6, but the anticipated and Gaussian plume model concentrations using the Briggs model are placed inside a factor of two with observed data of SF_6. Additionally, the suggested and Gaussian concentrations using the Briggs technique obtained 100% and 0.79% of the observed SF_6 concentrations, respectively, whereas the Gaussian concentrations using the Brookhaven method obtained 0.41% of the reported SF_6 concentrations.

With the reported concentrations of I^{135}, all of the Gaussian concentrations predicted by the Briggs and Brookhaven approaches fall within a factor of two. The observed concentrations of I^{135} yield 92%,!00%, and 100% of the suggested Gaussian concentrations using the Briggs and Brookhaven techniques, respectively. For every proposed Gaussian model, the NMSE and FB are around zero. The suggested Gaussian concentrations using the Brookhaven and Briggs methods have corresponding correlations of 0.93%, 0.62%, and 0.85%.

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