Computing support for testing equal values of the figurative numbers in the Pascal triangle

Miroslava Mihajlov Carević¹; Miloš Ilić²; Milena Petrović³; Nebojša Denić⁴

¹ Faculty of Mathematics and Computer Science, Alfa BK University, Beograd, Serbia,
miroslava.carevic.mihajlov@alfa.edu.rs

² Faculty of Mathematics and Computer Science, Alfa BK University, Beograd, Serbia,
milos.ilic@alfa.edu.rs

³ Faculty of Science and Mathematics, University of Priština, Kosovska Mitrovica, Serbia
milena.petrovic@pr.ac.rs

⁴ Faculty of Science and Mathematics, University of Priština, Kosovska Mitrovica, Serbia
nebojsa.denic@alfa.edu.rs

Abstract: In this paper we deal with a determination of numbers in a Pascal triangle that are simultaneously triangular, tetrahedral and hyper pyramidal, i.e. natural numbers $n, m, k \in \mathbb{N}$, such that it is

$$\binom{n}{2} = \binom{m}{3} = \binom{k}{4} \text{ for } n, m, k \in \mathbb{N} \text{ and } n \geq 2, m \geq 3, k \geq 4.$$

The collected results, obtained by mathematical analysis, were verified by computer. For this purpose, we used the C# programming language as well as the computer laboratory within our University in order to test the results. The results collected by computer confirmed the accuracy of the results obtained by mathematical analysis.

Keywords: Pascal triangle, computer support, figurative numbers.

1. Introduction

In a Pascal triangle consisting of binomial coefficients $\binom{n}{k}$ for $n, k \in \mathbb{N}$ and $k \leq n$, there is a presence of figurative numbers. The presence of triangular, tetrahedral and pentaedroid numbers is of particular interest to us. It is known that figurative numbers can be represented by a geometric representation with equally spaced points with each point representing a unit. Each figurative number is represented by a geometric object of the same type. Triangular numbers are represented by triangles, tetrahedral numbers by tetrahedrals, while pentaedroid numbers are hyper pyramidal

numbers in a four-dimensional space. A pentaedroid or hyper pyramid is a regular geometric body composed of 5 vertices, 10 edges, 10 surfaces, bounded by 5 tetrahedrons.

It is known that [1, 2, 3]

Triangular numbers: $1, 3, 6, 10, 15, 21, \dots, \frac{n(n+1)}{2}$

Tetrahedral numbers: $1, 4, 10, 20, 35, 56, \dots, \frac{n(n+1)(n+2)}{6}$

Pentaedroidni numbers: $1, 5, 15, 35, 70, 126, \dots, \frac{n(n+1)(n+2)(n+3)}{24}$

If we present Pascal's triangle as it follows:

```

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
. . .

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we can notice that the third column consists of a series of triangular numbers, the fourth column consists of a series of tetrahedral numbers, and the fifth column consists of a pentaedroid numbers.

It is also noticeable that the number 10 appears in a series of triangular and tetrahedral numbers. However, it does not appear in a series of pentaedroid numbers. The number 15 appears in a series of triangular and pentaedroid numbers but does not appear in a series of tetrahedral numbers. The number 35 appears in a series of tetrahedral and pentaedroid numbers but does not appear in a series of triangular numbers.

These findings raise the question of determining the numbers in the Pascal triangle that are simultaneously triangular, tetrahedral and pentaedroid, i.e. natural numbers $n, m, k \in \mathbb{N}$, such that it is

$$\binom{n}{2} = \binom{m}{3} = \binom{k}{4} \text{ za } n, m, k \in \mathbb{N} \text{ i } n \geq 2, m \geq 3, k \geq 4.$$

2. Materials and Methods

Equal values among figurative numbers and among binomial coefficients were the subject of research by scientists in the previous period. Determination of equal values for figurative numbers was done by Hajdu, Pintér, Tengely, Varga [4], for polygonal and pyramidal numbers by Brindza,

Pintér, Turjányi [5]. Weger [6], Pengelley [7], Guy [8], Pintér [9] and many other scientists have dealt with the equal values for binomial coefficients.

David Singmaster [10] showed that the only numbers that are simultaneously triangular and tetrahedral, that is, the numbers for which we have $\binom{n}{2} = \binom{m}{3}$ for $n, m \in \mathbb{N}$ and $n \geq 2, m \geq 3$, are follows : 1, 10, 120, 1540 i 7140. The first two numbers are trivial and the other numbers were found by a computer. In doing so, the calculated coefficients reached a value of up to 2^{48} . It is formed approximately $24 \cdot 10^6$ triangular numbers and $12 \cdot 10^4$ tetrahedral numbers.

If we were to solve an equation $\binom{n}{3} = \binom{m}{4}$ for $n, m \in \mathbb{N}$ and $n \geq 3, m \geq 4$, we could write it in form of:

$$\frac{n(n-1)(n-2)}{6} = \frac{m(m-1)(m-2)(m-3)(m-4)}{24} \quad (1)$$

By introducing a shift $a = n - 2$ and $b = \frac{m(m-3)}{2}$ equation (1) is reduced to

$$\frac{(a+2)(a+1)a}{6} = \frac{b(b+1)}{6} \quad \text{wherein } a, b \in \mathbb{N}$$

and that is the equation $a(a+1)(a+2) = b(b+1)$. (2)

Equation (2) was solved by Mordell [11] by introducing a shift $x = 2a + 2, y = 2b+1$ which brings us to the equation

$$\begin{aligned} \frac{x-2}{2} \cdot \left(\frac{x-2}{2} + 1 \right) \cdot \left(\frac{x-2}{2} + 2 \right) &= \frac{y-1}{2} \cdot \left(\frac{y-1}{2} + 1 \right) \\ \frac{x-2}{2} \cdot \frac{x}{2} \cdot \frac{x+2}{2} &= \frac{y-1}{2} \cdot \frac{y+1}{2} \quad \text{wherein } x, y \in \mathbb{N}. \end{aligned}$$

By arranging the expression on the left and right sides of the equality, we get the equation

$$x^3 - 4x = 2y^2 - 2$$

i.e. equation $2y^2 = x^3 - 4x + 2$. (3)

Mordel [11] was concerned with solving the Diophantine equation of form $Ey^2 = Ax^3 + Bx^2 + Cx + D$ where $A, B, C, D, E \in \mathbb{Z}$. He proved that this equation has only finitely many integer solutions when $B = 0$. Since $2y^2$ is always an even number, we conclude that x must be an even number. Therefore, the only solutions of equation (3) in set $\mathbb{N}$ are as follows:

$$x \in \{0, 2, 4, 12\}$$

i.e. $(x, y) \in \{(0, 1), (2, 1), (4, 5), (12, 29)\}$.

Based on this, we obtain solutions for the equation (2):

$$(a, b) \in \{(1, 2), (5, 14)\}$$

and finally, the solution for the equation (1):

$$(n, m) \in \{(3, 4), (7, 7)\}.$$

For $n = 3$ and $m = 4$ we obtain a trivial solution, number 1, that is, the first tetrahedral and the first pentaedroid number.

For $n = 7$ and $m = 7$ dobijamo broj 35:

$$\binom{7}{3} = \binom{7}{4} = 35.$$

Mordell [11] proved that (1, 2) and (5, 14) are the only solutions of equation (2) in the set of natural numbers. Thus, 1 and 35 represent the single natural numbers that are, simultaneously, tetrahedral and pentaedroid.

3. Results and Discussion

3.1. Results

Based on Singmaster results [10], the only numbers that are simultaneously triangular and tetrahedral, i.e. the numbers for which it is $\binom{n}{2} = \binom{m}{3}$ for $n, m \in \mathbb{N}$ and $n \geq 2, m \geq 3$, are following 1, 10, 120, 1540 and 7140. Mordell [11] proved that the only numbers that are simultaneously tetrahedral and pentaedroid, i.e. the numbers for which it is $\binom{m}{3} = \binom{k}{4}$ for $m, k \in \mathbb{N}$ and $m \geq 3, k \geq 4$ is following: 1 and 35. The only common element in these two sets of numbers is the number 1. Thus, the number 1 is the only number that is triangular, tetrahedral, and pentaedroid.

3.2. The computer search and discussion

In order to test the results, we used the C# programming language and the procedure described through the following algorithmic steps:

<p>Algoritam: The TTP algorithm for determination triangular, tetrahedral and pentahedroid numbers</p>
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<pre> int n = 0, ind, b1, m, b2, k, b3; do { ind = 0; do { n = n + 1; b1 = n * (n + 1) / 2; m = 0; do { </pre>
--

```
m = m + 1;
b2 = m * (m + 1) * (m + 2) / 6;
if (b2 == b1)
{
    Console.WriteLine("{0}. triangular number {1} = {2}. tetrahedral number {3}", n, b1, m, b2);
    ind = 1;
}
} while (b2 < b1 && ind == 0);
} while (ind == 0);
k = 0;
do
{
    k = k + 1;
    b3 = k * (k + 1) * (k + 2) * (k + 3) / 24;
    if (b3 == b2)
    {
        Console.WriteLine("= {0}. pentaedroid number {1}", k, b3);
    }
} while (b3 < b2);
} while (n < 10000000);
Console.WriteLine("Kraj");
Console.ReadKey();
```

The research was performed in the computer laboratory of Alfa BK University. We used Visual Studio 2017, an Intel i7-9700k processor and Windows 10 pro (64-bit) to test the program. The original intention was to test the program for all values of the variable $n < \text{maxint}$. As the value of the constant $\text{maxint} = 2\,147\,483\,647$ the calculations of values in the program did not end after 24 hours. Because of this, we have lowered the limit value and performed program testing for $n \leq 10000000$. We obtained the following outputs:

```
1. triangular number 1 = 1. tetrahedral number 1 = 1. pentaedrod number 1
4. triangular number 10 = 3. tetrahedral number 10
15. triangular number 120 = 8. tetrahedral number 120
55. triangular number 1540 = 20. tetrahedral number 1540
119. triangular number 7140 = 34. tetrahedral number 7140
```

Using computer search, we obtained only one number in the Pascal triangle that is triangular, tetrahedral, and pentaedroid. That is the number 1.

4. Main Text

In this paper, we have presented computer support for mathematical research and results. We have determined the binomial coefficients in the Pascal triangle which are simultaneously

triangular, tetrahedral and hyper pyramidal numbers. In order to test the obtained results, we wrote a program in a programming language C# whose execution was implemented in a computer laboratory of our University. The results collected by computer confirmed the accuracy of the results obtained by mathematical analysis.

5. Conclusions

Mathematical analysis showed that among the binomial coefficients in the Pascal triangle there is only one number that is simultaneously triangular, tetrahedral and hyper pyramidal. That is the number 1. It is also shown that there are 5 numbers in Pascal's triangle that are simultaneously triangular and tetrahedral. These are the numbers 1, 10, 120, 1540 and 7140. Verification of the obtained results by computer, confirmed their accuracy.

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