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On the Uses of LP and NLP to Formulate Game Theory Problems: A Brief Review

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Abstract. Game theory has been used for many years when making important strategical decisions. Many phenomena, specially in social and behavioural sciences are being modelled using game theory concepts. Game theory has seen its successful implementation in socio-economic and political decision making. From its beginning back in early 1920s, game theoretical models have been developed and used to explain many real-life scenarios. John Von Neumann formally introduced game theory during world war II back in 1944 along with fellow researcher Oskar Morgenstern. Since then, game theory has evolved in its own pace. In parallel, linear (LP) and nonlinear programming (NLP) have been crucial in many important research directions involving real life problems. The objective of this article is to review some published papers in literature that present many uses of LP and NLP formulation in game theoretical fields to aid in important strategic decisions.

Keywords. Review, Game Theory, LP, NLP, Behavioural science.

1. Introduction

The paper presents a summary of studies focusing on LP and NLP formulations of game theory. Some important journal articles are reviewed in this paper. The significant portions and highlights have been mentioned in Key findings section before conclusion. As indicated by the title, this review paper can be separated into two parts - LP formulation and NLP formulation of game theory.

In the first part, the LP formulation in two-person zero-sum games have been described in brief. In recent years, fuzzy LP models have been quite popular which has also been discussed briefly along with its solution procedures. An important study with (0-1) LP approach (also known as mixed binary) has been presented that correlates n-person normal game theory and its many applications to LP models [1].

In the second part, NLP approaches have been discussed to formulate game theoretical models. This kind of approach is new as compared to its LP counterpart. NLP formulation could be an important step in future research if it is pursued. However, in this paper, I-fuzzy numbers reward and n-person cooperative games have been illustrated with examples.

The unique contribution of this paper is that this is one of the very few attempts to illustrate the correlation between three of the most important fields in mathematics, namely game theory and LP/NLP programming. In the literature, there has not many papers that exactly describes this correlation and it could be an important step towards our better understanding of quantifying the strategic decision-making processes.

2. Literature review

This section serves as the main body for this paper. The journals being referred in the list of references' will be briefly discussed here. Only relevant contents will be focused ignoring minor aspects mentioned in those papers. The literature review as mentioned previously is divided into the following two subsections.

2.1. LP formulations in game theory

Two-person zero-sum games might just be the most common uses of LP in game theory [2]. A two-person zero-sum game is established [2] with only two players and the individual payoff for them is just the opposite of each other by sign (e.g. $+/-$). So, the total payoff for them would always be 0. The game matrix of such a game is illustrated in Table 1.

Table 1. Payoff matrix for two persons zero-sum games

Game Matrix
Strategy of player 2

		1	2	.	.	.	n
Strategy of player 1	1	a_{11}	a_{12}	.	.	.	a_{1n}
	2	a_{21}	a_{22}	.	.	.	a_{2n}
	.	.	.	.	.	.	.
	m	a_{m1}	a_{m2}	.	.	.	a_{mn}

The matrix elements namely a_{ij} $(i = 1, 2, 3 \ldots, m$ and $j = 1, 2, 3 \ldots, n)$ denotes the sum gained by player 1 if player 1 and 2 use strategies i and j simultaneously. Saddle point approach can be used to solve for pure strategies in such a formulation.

But LP formulation is extremely useful while these two-person zero-sum games deal with mixed strategies. To incorporate mixed strategies, it is generally assumed that each player plays each strategy with a certain probability. Let p_i and q_j be the probabilities for strategy i of the first player and strategy j of the second player. So, the following equations are equivalent to the game mentioned above:

$$\max \quad E = \sum_{j=1}^{n} \sum_{i=1}^{m} p_i q_j a_{ij}$$

$$\text{subject to} \quad \sum_{i=1}^{m} p_i = 1$$

$$\sum_{j=1}^{n} q_j = 1$$

$$p'.A \leq v$$

$$A.q' \geq v$$

$$0 \leq p_i \leq 1$$

$$0 \leq q_j \leq 1$$

Here, A=game matrix in Table 1 and v is the saddle point for the game (if any).

The objective function mentioned here denotes the payoff for player 1. Notice that the solution to this LP will determine the optimum strategies (mixed) with highest payoff for player 1 and lowest payoff for player 2. By formulating it the other way (e.g. minimize, instead of maximize E), the same solution can be reached intuitively [1]. But this time, the reward (e.g. payoff) will be maximized for player 2 and minimized for player 1.

Real life applications of LP in a farm production environment setting have been found in the literature [3]. In this special study, the author presented a minimax regret model to deal with several types of uncertainty like flood, drought while producing crops in the farm in a timely fashion. However, Laplace-Hurwicz model has also been successful to model these uncertainties. In brief, the maximin model focuses on the prediction of farmers' behaviour (satisfaction) towards the trade-off between their expected and actual payoff during natural calamities.

The following equations describes the LP formulation [3]:

$$\max \ V$$
subject to n constraints of the form:
$$\sum_{i=1}^{m} c_{ij} p_i - V \geq 0 \qquad (j = 1,2,\ldots,n)$$
$$\sum_{i=1}^{m} p_i \leq L$$
r constraints of the form
$$\sum_{i=1}^{m} a_{ik} p_i \leq b_k \qquad (k = 1, 2, \ldots, r)$$

Where, V, p_i, L, b, c_{ij} denote regret, activity level proportion, resources, and payoff (net) of the i^{th} policy adopted by the farmer and j^{th} state of nature at the same time. The formulation is intended to find out the optimal strategy for the farmer against the state of nature considering resource, land, and other relevant constraints.

Another interesting LP application is found in matrix games with interval-valued payoff [4]. This introduces uncertainty in the expected payoffs. In many real-life situations, the payoff might not be deterministic (a fixed value), rather stochastic (an interval/range) in nature. In this type of games, the interval payoff matrix for player 1 is as follows:

$$A = (A_{ij})_{m\times n}, \text{where } A_{ij} = [\underline{a}_{ij}, \overline{a}_{ij}]; (i = 1,2,\ldots,m; j = 1,2,\ldots,n)$$

It is to be noted that the value (v) of the total game will also be interval-valued due to the interval-valued payoffs for both players. v denotes the game value which can be expressed as $f(A_{ij})$ and thus indicates a non-decreasing payoff behaviour. The following brief formulation is presented [4]:

$$\max\{\bar{v}\}$$
$$\text{s.t.} \begin{cases} \sum_{i=1}^{m} \bar{a}_{ij} \bar{y}_i \geq \bar{v} \\ \sum_{i=1}^{m} \bar{y}_i = 1 \\ \bar{y}_i \geq 0 \\ \bar{v} \text{ unrestricted in sign} \end{cases}$$

Where, the payoff for player 1 (e.g. v) is considered and $\boldsymbol{y}$ represents the optimal strategy taken by player 1. As seen from the equations, this is a standard LP and can be easily solved with modified simplex algorithm to bring out exact solutions in terms of mixed strategies.

Mixed integer linear programming (MILP) is another technique that has been introduced in the literature [5]. It mostly deals with a finite game with n number of players that has only binary payoffs (e.g. 0 or 1). MILP has been used for many traditionally complex problems like facility layout optimization, production planning, etc.

A mixed strategy profile $\boldsymbol{x}^*$ (subset of all strategies set, X) is a Nash equilibrium of the n-person finite game $u^i(x^*) \geq u^i(x^i, x^{*-i})$ for all i in N where u^i is the utility function (expected payoff) for player 1 while using strategy 1. According to this definition, x^* is a Nash equilibrium if and only if there are two more parameters λ and μ with x^* and the following system of equations are satisfied [5]:

$$u^i\left(s_j^i, \mathrm{x}^{-i}\right) + \lambda_j^i - \mu_i = 0$$
$$e^{\mathrm{iT}} x^i - 1 = 0$$
$$x_j^i \lambda_j^i = 0, \mathrm{x}_j^i \geq 0,$$
$$\lambda_j^i \geq 0, \mathrm{j} = 1,2,\ldots,\mathrm{m}_i, \mathrm{i} = 1,2,\ldots,\mathrm{n}$$
$$\text{where } \mathrm{e}^i = (1,1,\ldots,1)^T \in R^{M_i}$$

There are detail proofs of the exact solutions found by the 0-1 mapping in the original paper. This algorithm (its possible extended forms) might be further used in other binary applications related to game theoretical models. But the authors admitted that more research or study will be required to find all mixed strategy equilibria for general purpose games.

Fuzzy LP approach is also introduced in the literature [9]. These games might be formulated with some existing level of vagueness in some or all the payoff matrix elements or given expected values of it. In this paper, the cumulative probability theory has been used a potential solution approach. The practical implementation of the fuzzy game is possible in many situations where precise data values are not found or difficult to obtain. The formulation is as follows:

$$\begin{aligned} &\max && \mathrm{cx} \\ &\text{subject to} && \tilde{a}_i x \leq \tilde{b}_i, \mathrm{i} \in \mathrm{M}, \\ &x \geq 0 \end{aligned}$$

The interval values are expected to follow fuzzy logics. Each player will be able to construct a personalized fuzzy LP model to solve for the solutions.

Along with these uses, an interesting two-sided matching is used frequently in our day to day life. Examples include, marriage market, college admission model, the assignment models etc. The stable marriage (or other related games) may be formulated as an assignment problem like the following [10]:

$$\text{maximize} \sum_{\mathrm{i,j}} \alpha_{\mathrm{ij}} . \mathrm{x}_{\mathrm{ij}}$$
$$\text{subject to} \sum_i x_{\mathrm{ij}} \leq 1$$
$$\sum_j x_{\mathrm{ij}} \leq 1$$
$$\mathrm{x}_{\mathrm{ij}} \geq 0$$

Here, the decision variables x_{ij} shows the probability that a marriage would form. The inequalities can be used to verify the basics of assignment problem where a mere partnership between two players (husband and wife) can be considered.

2.2. NLP formulations in game theory

Even though traditionally LP has been used much more than NLP in game theoretical models, recent progresses about NLP formulations have gained much interest among researchers. A very recent study [8] analyses the NLP formulation [9] approaches to deal with cooperative games involving n number of players with interval-valued payoffs. Those games have a core that can be described as follows [8]:

$$C(\hat{v}) = \left\{ \hat{x} \in R^n | \sum_{i \in N} \hat{x}_i =_I \hat{v}\,(N), \sum_{i \in S} \hat{x}_i \geq_I \hat{v}\,(S); \text{for} \forall S \subset N \right\}$$

The nonlinear approach to solve the core includes the following constraints

$$\begin{cases} \sum_{i \in S} \hat{x}_i \geq_I \hat{v}\,(S)(S \subset N) \\ \sum_{i \in N} \hat{x}_i =_I \hat{v}\,(N) \\ \bar{x}_i \geq \underline{x}_i (i = 1{,}2{,}....{,}\mathrm{n} \end{cases}$$

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The nonlinear programming model might be solved for cooperative games with satisfactory degrees required.

Along with interval-valued payoffs, fuzzy payoffs can also be modelled using NLP formulations [10]. I-fuzzy numbers payoffs matrix games has been approached with NLP and its parameterized variant.

$$\tilde{E}\,(\mathrm{x{,}y}) = x^T \tilde{F} y = \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{A}_{\mathrm{ij}}\, x_i y_j$$

While the payoff I (fuzzy number) is considered the gain (floor) can be found by:

$$\max\nolimits_{\mathrm{IF}}\{\hat{v}\}$$
$$\text{s.t.} \begin{cases} \sum_{i=1}^{m} \tilde{A}_{\mathrm{ij}} x_i \geq_{\mathrm{IF}} \hat{v}(j = 1{,}2{,}....{,}\mathrm{n}) \\ x_1 + x_2 + x_3 + \ldots\ldots + x_m = 1 \\ x_i \geq 0 (i = 1{,}2{,}...{,}\mathrm{m}) \end{cases}$$

Similarly, the gain for the other player can be calculated by incorporating both formulations. Thus the overall problem becomes nonlinear like the following:

$$\max\{\eta(\hat{v},\lambda)\}$$
$$\text{s.t}\begin{cases}\sum_{i=1}^{m}\eta(\tilde{A}_{\text{ij}},\lambda)x_i \geq \eta(\tilde{v},\lambda) & (j = 1,2,...,\text{n})\\ x_1 + x_2 + ... + x_m = 1 & \\ x_i \geq 0 & (i = 1,2,...\text{m})\end{cases}$$

Then, the mixed strategy for the other one can also be formulated and solved using nonlinear programming approach.

An adaptive iterative scheme involving dynamic programming was proposed for solving unknown nonlinear zero-sum game based on online data [11]. This method is implemented in control systems to solve Hamilton-Jacobi-Issacs (HJI) equation. It is difficult because of nonlinearity. An NL zero-sum game (ZSG) is taken as a medium to find the feedback control signal described by the following:

$$V^*(x) \equiv \min\max \text{J(x,u,}\omega)$$

which is strictly subjected to the following:

$$\min\max \text{J(x,u,}\omega) = \max\min \text{J(x,u,}\omega)$$

3. Key findings

This article discusses the success of LP formulation in game theory. Many strategic games can be modelled using LP formulation. Another important insight is that the requirement for deterministic reward for LP formulation is not necessary. The rewards (e.g. payoff) can be stochastic in nature and can still allow LP formulation in many problems. Payoffs expressed with (fuzzy) numbers can also be tried out as well.

There are lots of variation in game theoretical models that are not discussed in this paper. The formulations presented here might only be used for some specific versions of game theoretical model as mentioned. This paper does not intend to analyze any socio-economic, political behavior or related issues associated with game theoretical models.

Game theory and operations research tools are interrelated while making complex decisions. But a generalized framework to use LP/NLP game theoretical models need further investigation by the researchers. But the uniqueness of a game and its different variants should be understood and handled properly while modelling.

4. Conclusion

Complex nature of games might be very difficult to formulate or solve in real life. A more sophisticated approach might be carried out on the solutions of complex games only which would drastically decrease the complexity and make room to apply some relaxation techniques just like we do in the case of mathematical formulation of production problems. When there are lots of active players participating in a game, it becomes increasingly difficult to balance out the payoff for everyone.

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drastically decrease the complexity and make room to apply some relaxation techniques just like we do in the case of mathematical formulation of production problems.

In this paper, concepts of zero-sum games and its similarity in LP formulation have been discussed. It would be a great advanced if a generalized framework of LP/NLP can be discovered for game theoretical models. The special cases of many problems mentioned in this paper are good indicators for the need of a generalized framework in future. However, this paper is restricted to the uses of LP an NLP in game theory. There are other optimization techniques involved in real life situations which are out of scope for this paper.

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